A note on the breakdown of continuity in the motion of a compressible fluid

By D. C. PACK
Department of Mathematics, The Royal College of Science and Technology, Glasgow

(Received 1 August 1959)

By a consideration of the relationships holding along the characteristics in an unsteady motion involving plane, axially or spherically symmetrical flow of compressible inviscid fluid, it is shown that the existence of a region of compression anywhere in the flow must lead eventually to the breakdown of continuity. The paper generalizes and unites previous work on this topic, and discusses some recent numerical calculations in which the expected discontinuity was not found.

1. Introduction

The fact that a wave of compression, in one-dimensional unsteady flow of an inviscid non-heat-conducting gas always leads eventually to a breakdown in the continuity of the flow has been known for a century. Challis (1848) observed that Poisson's simple-wave solution (1808) of the differential equation of flow in an isothermal gas could not always be solved uniquely for the velocity, and it was Stokes (1848) who first attempted to insert a discontinuity of velocity into the flow in order to continue it beyond the point in (x, t)-space at which the mathematical solution broke down. Although it was many years before the exact nature of these discontinuities, or 'shock waves', was correctly worked out, it is now well understood that in the simple wave resulting from the positive acceleration of a piston into a (one-dimensional) gas at rest, a shock wave must occur (see, for example, Courant & Friedrichs 1948). It is not, however, immediately obvious that such a breakdown of continuity in a compressive wave must necessarily occur when the motion has cylindrical or spherical symmetry. In these cases, the rapid increase in the volume occupied by a gas during its expansion may cause the pressure gradients in the gas to fall in the initial stages. A calculation performed by Unwin (1941) on the expansion of a spherical mass of gas released suddenly from a state of rest did not reveal the occurrence of a discontinuity up to the t-coordinate (t being the time) at which he ended his computations. In a recent paper Fox & Ralston (1957) have reworked Unwin's example, and while disagreeing with his numerical results, have concluded, even more strongly than he did, that no shock wave will result from a continuation of their calculations. Roberts (1957), calculating the development of a stronger initial distribution of pressure, also reached these conclusions.

In contrast with these results, it was shown by Hantzsche & Wendt (1940), who examined the development of a wave moving irrotationally with a region of compression at its head into gas at rest, that ultimately the continuity would break down. Kuo (1947) has discussed a similar type of flow and has also found some sufficient conditions for breakdown in the general case by a consideration of the flow in the hodograph plane.

Furthermore (although these authors seem not to have been aware of it) Burton showed, as long ago as 1893, that the continuity of an expanding spherical wave will eventually break down if there is at all times a region of compression, no matter how weak, somewhere within the wave. Burton obtained his results by intuitive arguments based upon inequalities.

In this note the problem is reconsidered in terms of the theory of characteristics, which enables Burton's arguments to be expressed in a more precise form and at the same time generalizes the results of Hantzsche & Wendt.

2. Equations of motion

Let us consider the characteristic surfaces of a gas motion, these surfaces being given by the equations

$$\frac{dr}{dt} = u \pm c,$$

where u is the particle velocity and c the local speed of sound at a point (r, t), r being distance from the centre, from the axis or from a fixed plane according as the motion is radially or cylindrically symmetrical or plane (one-dimensional). The conditions of compatibility, which have to be satisfied on these surfaces are, in differential form,

$$du \pm \frac{2dc}{\gamma - 1} \pm \frac{\epsilon uc}{r} dt - \frac{1}{\gamma} c^2 \frac{\partial \Sigma}{\partial r} dt = 0,$$

where γ is the (supposed constant) ratio of specific heats; $\epsilon = 2$, 1 or 0 according as we are considering radially or cylindrically symmetrical or plane flow; Σ is a function of the specific entropy S defined by $\Sigma(S) = \{2\gamma/(\gamma-1)\}\log c - \log p$, p being the pressure; and where the alternative signs correspond respectively to those in the previous equation of the characteristics. We shall also suppose that the two families of characteristics in the (r, t)-plane are respectively designated in some way as the curves $\alpha =$ const. and $\beta =$ const. We then consider two points P and Q which are allowed to vary in such a way that, in (α, β)-space, their co-ordinates are (α_1, β) and (α_2, β), where β is a variable and α_1, α_2 are arbitrary constants (see figure 1).

Suppose that P is at radial distance r and Q at distance $r + \Delta r$. We shall investigate the change in Δr as the time increases and P, Q move along their α-characteristics. If at some time Δr becomes zero, the α-characteristics will intersect, an indication of the breakdown of the continuity of the motion.

If such a breakdown occurs in the early stages of the motion, there is nothing to prove. The important question is whether, if no discontinuity occurs before the disturbances become of small magnitude, the motion can avoid indefinitely the onset of such a discontinuity. We suppose, therefore, that P and Q lie on a β-characteristic which is sufficiently far from the origin for Δr to be small compared with r. Also, we suppose that the disturbances have decayed to such an extent that u is small compared with c; on the other hand, Δu and Δc (the symbol Δ represents the increment in the value of the following quantity from the point P

to the point Q), while small, may be of the same order of magnitude as u. Since both P and Q move on α-characteristics, it follows that

$$\frac{\partial}{\partial\beta}\left(\Delta u+\frac{2\Delta c}{\gamma-1}\right)+\epsilon\Delta\left(\frac{uc}{r}\frac{\partial t}{\partial\beta}\right)=\frac{1}{\gamma}\Delta\left(c^2\frac{\partial\Sigma}{\partial r}\frac{\partial t}{\partial\beta}\right).$$

When the disturbances are weak, $\Sigma(S)$ is of the order of magnitude of $(u/c)^3$ at most, the order of magnitude obtained when there is already a weak shock-wave ahead of the region under consideration; in homentropic flow, $\Sigma(S)$ is constant.

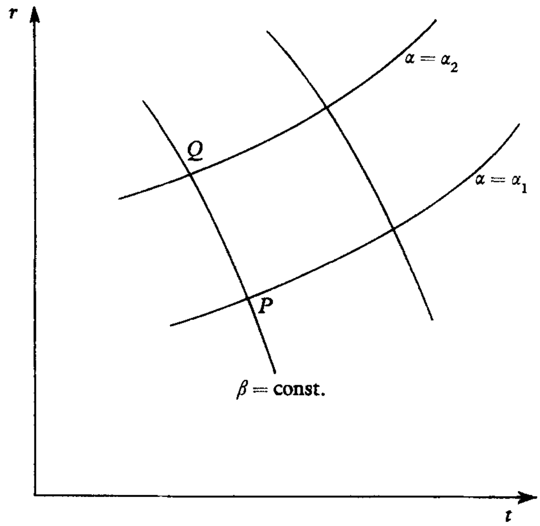

FIGURE 1.

Hence, when terms of second and higher orders of smallness are neglected in the above equation, we obtain

$$\frac{\partial}{\partial\beta}\left(\Delta u+\frac{2\Delta c}{\gamma-1}\right)+\epsilon c_0\Delta\left(\frac{u}{r}\frac{\partial t}{\partial\beta}\right)=0, \tag{1}$$

where c_0 is the speed of sound in the gas at rest.

Since P and Q lie on a β-characteristic, we have

$$\Delta u-\frac{2\Delta c}{\gamma-1}-\int_P^Q\frac{\epsilon uc}{r}\frac{dr}{u-c}=0,$$

the integral being taken along the characteristic and the entropy term being omitted for the same reason as in (1). An examination shows that the third term above may also be neglected, to the same order of approximation. This leads to the simple relation

$$\Delta u=2\Delta c/(\gamma-1). \tag{2}$$

If we use a suffix zero to denote the conditions at some time $t=T$ for which r is already sufficiently large (and equal to r_0), then we may replace r by $r_0+c_0(t-T)$ to a first approximation and rewrite the equation (1) above, with the aid of (2), as

$$\frac{\partial}{\partial\beta}(\Delta u)+\tfrac{1}{2}\epsilon c_0\left(\frac{1}{c_0 r}\frac{\partial r}{\partial\beta}\Delta u\right)=0.$$

it is typically found in the theories that the pressure coefficient* C_{p_s} at the separation point roughly obeys the relation $C_{p_s} \propto (T_w/T_{wz})^{-n}$, where n is between 0·5 and 1, and T_w is measured on the absolute scale. This considerable predicted effect of heat transfer is of great interest, because in many practical applications where boundary layer separation at supersonic speeds may occur the wall temperature will be much lower than the zero heat transfer value. Also, in certain wind tunnel investigations observations

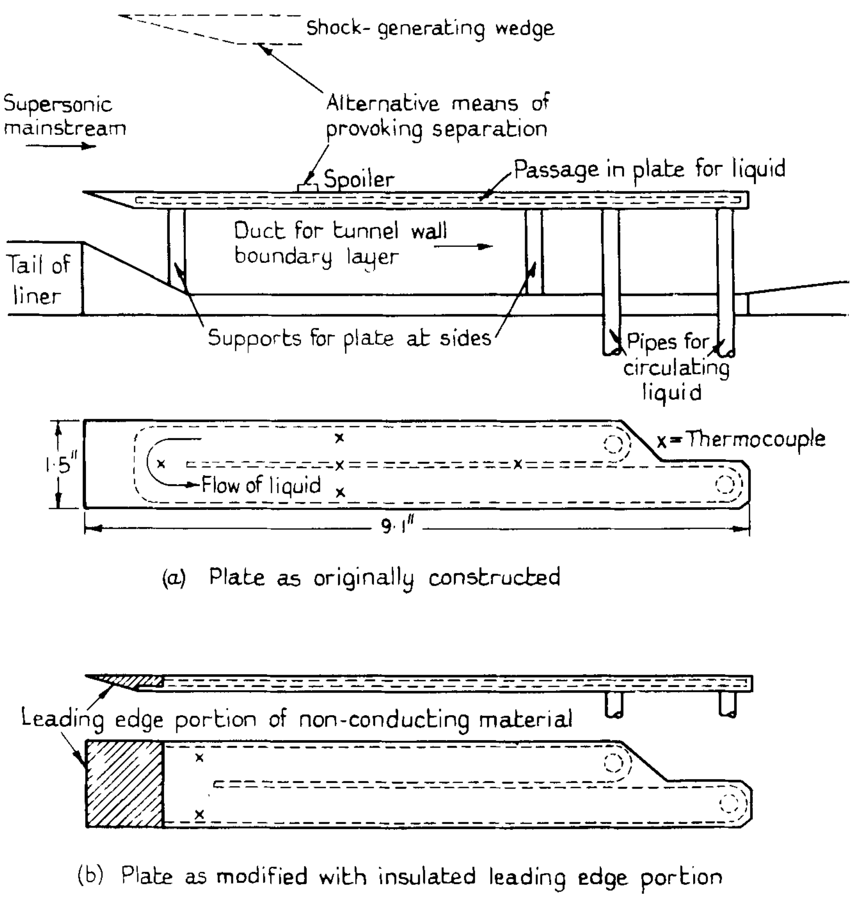

Figure 1. The flat plate on which the test boundary layer was formed.

are made before the model and flow are in thermal equilibrium, so that errors may arise if the effects of heat transfer on separation are appreciable. Previous experimental work (for example Gadd, Holder & Regan 1954) on separation in supersonic flow has all been concerned with the insulated condition. Hence it was decided to investigate experimentally cases of laminar and turbulent separation with the wall heated or cooled.

* The pressure coefficient $C_p = 2(p-p_1)/\gamma M_1^2 p_1$ where p = pressure, p_1 = free-stream pressure, γ = ratio of specific heats, and M_1 = free-stream Mach number.

$(du/dr)_\beta$ may at first diminish on account of the factors $r^{-\frac{1}{2}}$ and r^{-1} which occur in the right-hand sides of the formulae given above; only if k is sufficiently large will $(du/dr)_\beta$ increase continuously with time. In both cases, however, it is clear that the denominator vanishes eventually, for any $\gamma > -1$, so that an infinite gradient of velocity must appear on the characteristic. For spherically symmetrical motion, for example, it will occur not later than time t given by

$$t-T = (r_0/c_0)\,[\exp[2c_0/r_0k(\gamma+1)]-1].$$

Hence, for any plane, cylindrically or spherically symmetrical gas motion the continuity must eventually break down along a characteristic on which the motion is compressive.*

When $\gamma = -1$, there is no shock wave. For plane waves $(du/dr)_\beta$ is constant in this case. This is in agreement with the known result that, for $\gamma = -1$, plane waves propagate without change of form.

4. Comparison with some computations

Fox & Ralston (1957) and Roberts (1957) have carried out numerical calculations for a gas initially at rest with a density distribution given by

$$\rho = \rho_0[1+p\exp(-4r'^2)],$$

where $r' = r/a$ (a is some characteristic length, ρ_0 the undisturbed density of air) and $p = 2$ (Fox & Ralston, following Unwin, 1941), or $p = 5$ (Roberts). They did not find any evidence of a breakdown of continuity. The disturbances represented by these distributions quickly become small everywhere, and when the theory of the present paper is applied directly to them, with $T = 0$, it is found that a shock wave should begin at $t' = 0{\cdot}7$ and $t' = 0{\cdot}3$, where $t' = c_0t/a$, corresponding to $r' = 1{\cdot}2$ and $r' = 0{\cdot}9$, respectively.

In discussing this difference between the theoretical conclusions and those found by calculation, it may be said that, for spherical waves, the shock intensities can be so small that they may be masked by the errors involved in numerical calculations when the partial differential equations are replaced by finite difference calculations. The tendency to form extremely weak shock-waves in spherical expansions was indicated by Taylor in (1946) and Lighthill showed analytically in 1948 that, in the flow resulting from the expansion of a spherical piston with Mach number 1/5, the leading shock wave was of order e^{-44} in strength, i.e. 'of an order of smallness rarely encountered in physical problems'. When one considers this in conjunction with the fact that Roberts's finite difference system introduced an artificial viscosity which would have the effect of smoothing out discontinuities (as he himself says), it may be correct to conclude that the discrepancy between the conclusions of this paper and the results of the calculations which Roberts carried out is more apparent than real; that shock waves of the strength likely to occur could not have been detected by the numerical method adopted.

* It may be noted that Burton pointed out in his original paper that the inevitability of a discontinuous motion would not apply to flow in dimensions greater than three.

Fox & Ralston computed the case $p = 2$ by means of a finite difference scheme for the characteristics and the equations of compatibility. They remark that 'any tendency for the characteristics to converge is quickly dispelled and so there is never any sign of the formation of a shock'. As the author (1948) has shown in a paper on the formation of shock waves in jets, the initial point of a shock wave may be found by constructing characteristics over a field sufficient to allow mutual intersections of several neighbouring characteristics of the same family, and by extrapolating back to find the point at which two infinitesimally separated characteristics first begin to form the envelope of their family. The point $r' = 1{\cdot}2$, $t' = 0{\cdot}7$ lies near to the boundary of the region computed by Fox & Ralston, so that here again, with a very weak shock, the breakdown could not be expected to reveal itself.

The author therefore believes that the computations so far carried out could not have revealed the shock wave which was sought.

REFERENCES

Burton, C. V. 1893 *Phil. Mag.* (5), **35**, 317.

Challis, J. 1848 *Phil. Mag.* (3), **32**, 494.

Courant, R. & Friedrichs, K. O. 1948 *Supersonic Flow and Shock Waves.* New York: Interscience Publishers.

Durand, W. F. (Ed.) 1935 *Aerodynamic Theory*, **3**, 216. Berlin: Springer.

Fox, P. & Ralston, A. 1957 *J. Maths. Phys.* **36**, 313.

Hantzsche, W. & Wendt, H. 1940 *Jb. dtsch. Luftfahrtf.*

Kuo, Y-H. 1947 *Quart. Appl. Maths.* **4**, 349.

Lighthill, M. J. 1948 *Quart. J. Math. Appl. Mech.* **1**, 309.

Pack, D. C. 1948 *Quart. J. Math. Appl. Mech.* **1**, 1.

Poisson, S. D. 1808 *J. Éc. polyt., Paris*, **7**, 319.

Roberts, L. 1957 *J. Maths. Phys.* **36**, 329.

Stokes, E. E. 1848 *Phil. Mag.* (3), **33**, 349.

Taylor, Sir Geoffrey 1946 *Proc. Roy. Soc.* A, **186**, 273.

Unwin, J. J. 1941 *Proc. Roy. Soc.* A, **178**, 153.